**Prop:** Given a parametrization $\Sigma$ with $T_pS = \text{span}\{\Sigma_{u_1}, \Sigma_{u_2}\}$, the shape operator $S = -dN_p : T_pS \longrightarrow T_pS$ is given by the matrix:

$$\boxed{S = (g_{ij})^{-1}(A_{ij})} \quad \text{———} \quad (\#)$$

**Proof:** By definition,

$$\begin{cases} S\left(\frac{\partial}{\partial u_1}\right) = -dN\left(\frac{\partial}{\partial u_1}\right) = -\frac{\partial N}{\partial u_1} = a\frac{\partial}{\partial u_1} + b\frac{\partial}{\partial u_2} \\ S\left(\frac{\partial}{\partial u_2}\right) = -dN\left(\frac{\partial}{\partial u_2}\right) = -\frac{\partial N}{\partial u_2} = c\frac{\partial}{\partial u_1} + d\frac{\partial}{\partial u_2} \end{cases}$$

Written as matrices,

$$\begin{pmatrix} | & | \\ -\frac{\partial N}{\partial u_1} & -\frac{\partial N}{\partial u_2} \\ | & | \end{pmatrix} = \begin{pmatrix} | & | \\ \frac{\partial}{\partial u_1} & \frac{\partial}{\partial u_2} \\ | & | \end{pmatrix} \begin{pmatrix} a & c \\ b & d \end{pmatrix}$$

$$\Rightarrow \underbrace{\begin{pmatrix} - & \frac{\partial}{\partial u_1} & - \\ - & \frac{\partial}{\partial u_2} & - \end{pmatrix} \begin{pmatrix} | & | \\ -\frac{\partial N}{\partial u_1} & -\frac{\partial N}{\partial u_2} \\ | & | \end{pmatrix}}_{(A_{ij})} = \underbrace{\begin{pmatrix} - & \frac{\partial}{\partial u_1} & - \\ - & \frac{\partial}{\partial u_2} & - \end{pmatrix} \begin{pmatrix} | & | \\ \frac{\partial}{\partial u_1} & \frac{\partial}{\partial u_2} \\ | & | \end{pmatrix}}_{(g_{ij})} \begin{pmatrix} a & c \\ b & d \end{pmatrix}$$

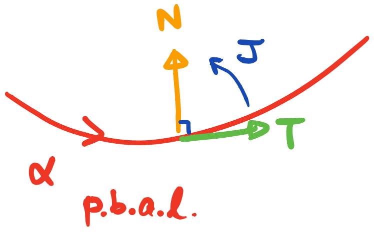
Multiplying $(g_{ij})^{-1}$ yields $(\#)$.

$\blacksquare$

## § Normal curvatures (do Carmo § 3.2)

We now want to interpret the $2^{nd}$ f.f. $A$ as evaluating the curvature of certain plane curves lying on $S$.

Recall:



$$\boxed{k = \langle \alpha'', N \rangle}$$

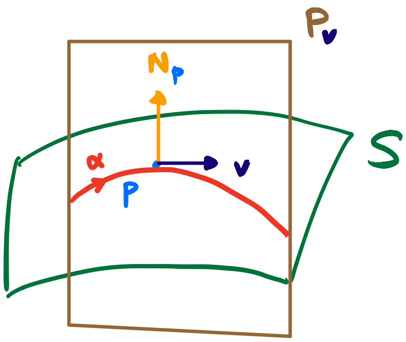
with orientation $\{T, N\}$

Let $S \subseteq \mathbb{R}^3$ be a surface oriented by $N$.

$$\boxed{\text{Fix } p \in S \text{ and a unit tangent vector } v \in T_pS}$$

Consider the oriented plane

$$P_v = \text{span}\underbrace{\{v, N_p\}}_{\text{pos. orientation}}$$

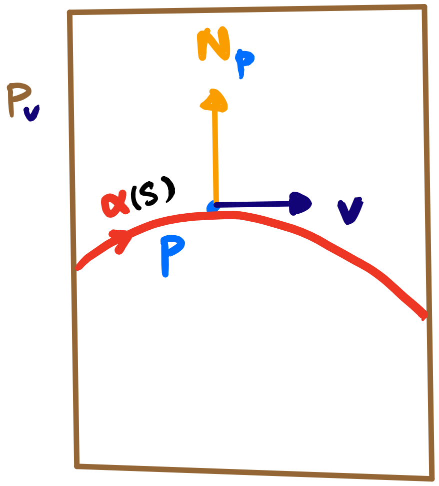


which cuts $S$ along some regular curve (why?) p.b.a.l.

$$\alpha : (-\varepsilon, \varepsilon) \longrightarrow S \quad \text{s.t.} \quad \alpha(0) = p, \quad \alpha'(0) = v$$

which can also be regarded as a plane curve on $P_v$ with curvature

$$\boxed{k_v = \langle \alpha''(0), N_p \rangle} \quad \text{— } (\#)$$



On the other hand, since $\alpha \subseteq S$

$$\Rightarrow \quad \alpha'(s) \in T_{\alpha(s)}S \;, \; \forall s$$

$$\Rightarrow \quad \langle \alpha'(s), N(\alpha(s)) \rangle \equiv 0 \quad \forall s$$

Differentiate
w.r.t. $s$
at $s=0$

$$\Longrightarrow \quad \underbrace{\langle \alpha''(0), N_p \rangle}_{k_v} + \underbrace{\langle \alpha'(0), dN_p(\alpha'(0)) \rangle}_{-A(v,v)} = 0$$

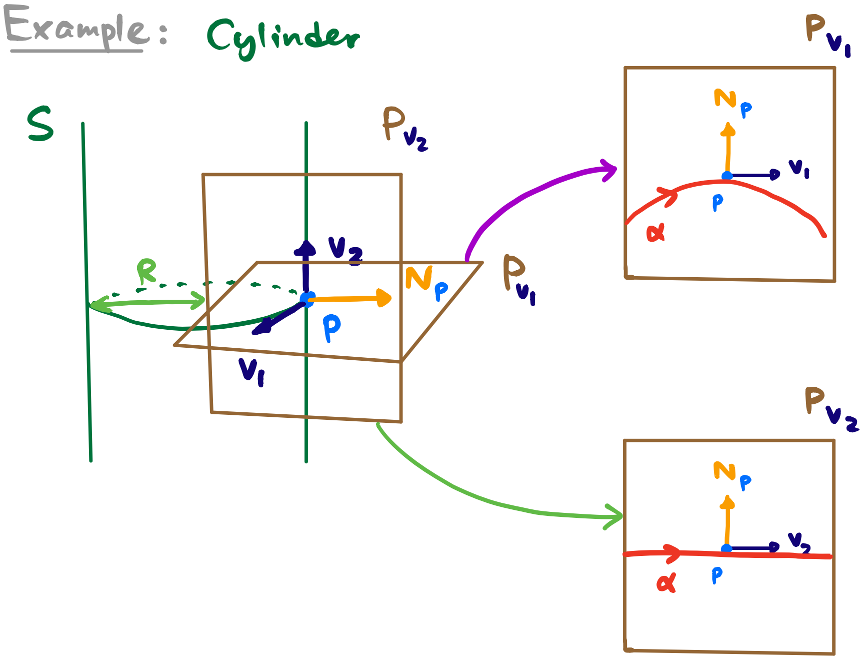
i.e. $A(v,v) = k_v$ (normal curvature along $v$)

By the variational characterization of eigenvalues,

the principal curvatures (at $p$) are

$$\boxed{\kappa_1 = \min_{\substack{v \in T_pS \\ \|v\|=1}} k_v} \quad \& \quad \boxed{\kappa_2 = \max_{\substack{v \in T_pS \\ \|v\|=1}} k_v}$$

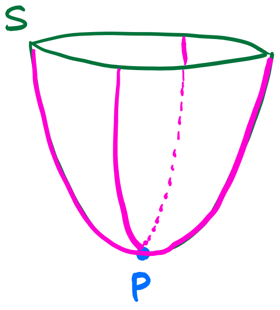
**Example:** Cylinder



$$k_{v_1} = -\frac{1}{R}$$

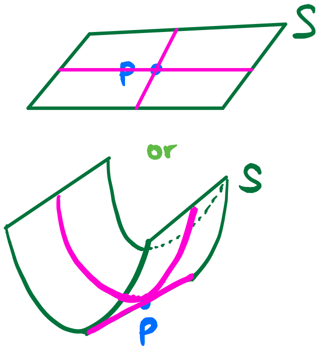
$$k_{v_2} = 0$$

We have the following local picture of surfaces:



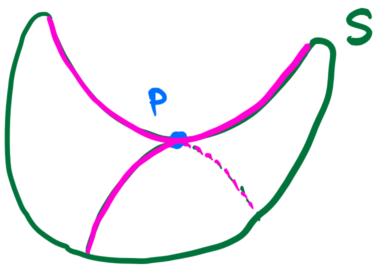
$K > 0$

"elliptic"



$K = 0$

"planar / parabolic"



$K < 0$

"hyperbolic"

## § Totally Umbilic Surfaces (do Carmo § 3.2)

**Thm:** $S \subseteq \mathbb{R}^3$ connected totally umbilic $\Longrightarrow$ $S$ is contained in a plane or sphere.

**Proof:** Recall that totally umbilic means

$$\kappa_1(p) = \kappa_2(p) \quad \text{at every } p \in S$$

i.e. $\exists$ function $f : S \to \mathbb{R}$ s.t.

$$S = -dN_p = f(p)\,\mathrm{Id} : T_pS \to T_pS$$

**Ex:** Show that $f$ is smooth!

For any parametrization $\mathbf{X}(u,v)$ on $S$,

$$\begin{cases} S\left(\dfrac{\partial \mathbf{X}}{\partial u}\right) = f\dfrac{\partial \mathbf{X}}{\partial u} \\ S\left(\dfrac{\partial \mathbf{X}}{\partial v}\right) = f\dfrac{\partial \mathbf{X}}{\partial v} \end{cases} \Rightarrow \begin{cases} -\dfrac{\partial N}{\partial u} = f\dfrac{\partial \mathbf{X}}{\partial u} \\ -\dfrac{\partial N}{\partial v} = f\dfrac{\partial \mathbf{X}}{\partial v} \end{cases} \qquad (*)$$

$$\Rightarrow \begin{cases} -\dfrac{\partial^2 N}{\partial v \partial u} = \dfrac{\partial f}{\partial v}\dfrac{\partial \mathbf{X}}{\partial u} + f\dfrac{\partial^2 \mathbf{X}}{\partial v \partial u} \\ -\dfrac{\partial^2 N}{\partial u \partial v} = \dfrac{\partial f}{\partial u}\dfrac{\partial \mathbf{X}}{\partial v} + f\dfrac{\partial^2 \mathbf{X}}{\partial u \partial v} \end{cases}$$

$$\Rightarrow \qquad \frac{\partial f}{\partial v}\frac{\partial \mathbf{X}}{\partial u} = \frac{\partial f}{\partial u}\frac{\partial \mathbf{X}}{\partial v}$$

$$\Rightarrow \qquad \frac{\partial f}{\partial v} = \frac{\partial f}{\partial u} \equiv 0 \qquad \left(\because \left\{\frac{\partial \mathbf{X}}{\partial u}, \frac{\partial \mathbf{X}}{\partial v}\right\} \text{ lin. indep.}\right)$$

i.e. $f$ is (locally) constant ($\because$ $S$ connected)

Case 1 : $\boxed{f \equiv 0}$ $\Rightarrow$ $N \equiv$ const. plane!

Case 2 : $\boxed{f \equiv c \neq 0}$

Claim: $S$ is contained in a sphere of radius $\frac{1}{|c|}$

It suffices to show:

$$\mathbf{X} + \frac{1}{f} N \equiv \text{const. } p_0 \leftarrow \text{center of the sphere}$$

Note that:

$$\frac{\partial}{\partial u}\left(\mathbf{X} + \frac{1}{f} N\right) = \frac{\partial \mathbf{X}}{\partial u} + \frac{1}{f}\frac{\partial N}{\partial u} \overset{(*)}{\equiv} 0 .$$

Similarly,

$$\frac{\partial}{\partial u}\left(\mathbf{X} + \frac{1}{f} N\right) = 0 .$$

This proves the claim since $S$ is connected.

$\square$

By Spectral Theorem, the self adjoint operator

$$S = -dN_p : T_pS \longrightarrow T_pS \quad \text{diagonalizable}$$

eigenvalues : $\kappa_1$ , $\kappa_2$ principal curvatures

eigenvectors (unit) : $v_1$ , $v_2$ principal directions

(at $p$)

<u>Def$^{\underline{n}}$</u>: $p \in S$ is an umbilic point $\Longleftrightarrow$ $\boxed{\kappa_1 = \kappa_2}$ at $p$

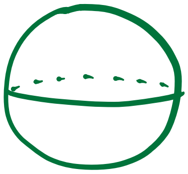
$S$ is totally umbilic $\Longleftrightarrow$ every $p \in S$ is umbilic.

<u>Examples</u>:

<u>Plane</u>

$\kappa_1 = \kappa_2 \equiv 0$

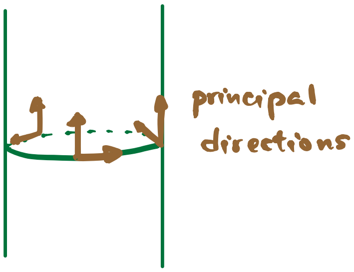
<u>Sphere</u>



$\kappa_1 = \kappa_2 \equiv -\frac{1}{R}$

totally umbilic

<u>Cylinder</u>



$\kappa_1 = -\frac{1}{R}$ $\quad \kappa_2 = 0$

not umbilic